Chapter 1

Differential and pseudodifferential operators on vector bundles

1.1 Differential operators on manifolds

1.1.1 What is manifold?

Etymologically, manifold means a collection of maps. In fact, it really is.

For thousands of years, people have never stopped exploring the land we live on.

They explore the earth and draw the picture to note down everything they see, which is called the map. The idea of drawing the map to record the roads is not hard, even an old horse could do it in its brain. But usually one map is not enough. For example, if we want to drive the car from Pudong airport to ECNU, we would be disappointed to see that, not like our neighborhood, we could not find ECNU in the screen map containing the airport.

So we have to glue another map together to find the road we need.

In order to glue the maps together, the only necessary condition is that **in the intersection area, a continuous road in one map must continuous in another one.** Otherwise we will get lost. (Of course, due to the modern technologies, AI could do this for us even with a sweet voice.)

On the other hand, the fact that earth is a round ball obviously also gives the fundamental obstruction to describe everything in one map. We need to draw the map piece by piece and glue them together.

However, it is unbelievable and a miracle that we can get the converse: from the maps with scales and the gluing methods we can see that our earth is a ball, not a torus, by the Gauss-Bonnet theorem. (If the reader knows the Gauss-Bonnet theorem before, which we will not mention in the following sections, please try to explain it.)

Let us explain a bit the importance of this genius idea. For the earth, the earth is a round ball is not news. We could work hard and earn enough money to take the spacecraft to the moon to confirm that the earth is really round, does not has a hole and not like a cup.

But for the space-age now, we want to explore the universe. It is not easy to decide that the universe has a hole or not. To be honest, we are not the God. We cannot see the universe outside it like the people on the moon. The only method is to search it, draw the 3-dimensional map everywhere, glue them together and try to obtain the global properties by the local explorations.

For our mathematicians, we are used to abstract the ideas and put all dimensions together. We define the manifold as the topological spaces which can be studied in this process. Let us end the story and start to study the math.

Definition 1.1.1. Let M be a Hausdorff and second countable topological space. We say that M is a **topological manifold** of dimension n if every point $x \in M$ has an open neighborhood, which is homeomorphic to an open set in $\mathbb{R}^n$.

For topological manifold M, there exists a open cover $\{U_i\}_i$ of M such that each U_i has a map $\phi_i : U_i \to \mathbb{R}^n$ and $\phi_i : U_i \to \phi(U_i) \subset \mathbb{R}^n$ is a homeomorphism. The pair (U_i, ϕ_i) is called a chart and the collection $\{(U_i, \phi_i)\}_i$ is called an atlas.

1.1.2 How to do analysis on manifold?

Since M is a topological space, we could study a continuous function $f : M \to \mathbb{R}$. But if we want to apply the achievements of human after Newton on manifolds, we have to find a way to take the derivative of f. **We don't know how to do analysis on manifold, but we know how to do it on $\mathbb{R}^n$.**

The most natural idea is to take the derivative on each chart (U_i, ϕ_i). If

$$f \circ \phi_i^{-1} : \phi_i(U_i) \subset \mathbb{R}^n \to \mathbb{R} \tag{1.1.1}$$

is smooth, we could take the derivative of $f \circ \phi_i^{-1}$ as the derivative of f. However, if $U_i \cap U_j \neq \emptyset$, for $x \in U_i \cap U_j$, even if $f \circ \phi_i^{-1}$ is smooth at x, $f \circ \phi_j^{-1}$ may be not differentiable at x. If we want to make

$$f \circ \phi_j^{-1} = (f \circ \phi_i^{-1}) \circ (\phi_i \circ \phi_j^{-1}) \tag{1.1.2}$$

smooth at x, We need to assume further that

$$\phi_{ij} := \phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \subset \mathbb{R}^n \to \phi_i(U_i \cap U_j) \subset \mathbb{R}^n \tag{1.1.3}$$

is smooth. If for any i, j such that $U_i \cap U_j \neq \emptyset$, (1.1.3) is smooth, the statement "f is smooth at $x \in M$" is meaningful.

Remark that $f(x) \equiv 0$ is a smooth map.

Definition 1.1.2. A smooth atlas on a topological manifold is an atlas $\{(U_i, \phi_i)\}_i$ such that for any i, j such that $U_i \cap U_j \neq \emptyset$, (1.1.3) is smooth. We say two smooth atlases are equivalent if they determine the same collection of smooth functions on M. The equivalent class of smooth atlases is called a smooth structure. A topological manifold with a smooth structure is called a smooth manifold.

The main purpose of this definition is to establish a home for the smooth function on manifold.

Furthermore, we could define a smooth map between two smooth manifolds.

Definition 1.1.3. Let M and N be two smooth manifolds with smooth atlases $\{(U_i, \phi_i)\}_i$ and $\{(V_j, \varphi_j)\}_j$. Let $f : M \to N$ be a continuous map. For $m \in U_i \subset M$, we say that f is smooth at m if for any V_j containing $f(m)$, $\varphi_j \circ f \circ \phi_i^{-1}$ is smooth at $\phi_i(m)$. If f is smooth at any $m \in M$, we say that f is a smooth map.

If the smooth map f has an inverse, which is also smooth, we say f is a diffeomorphism. For example, ϕ_{ij} in (1.1.3) is a diffeomorphism.

In this note, we will work in the $\mathscr{C}^\infty$ category. In the following sections, if we say "a manifold", we mean "a smooth manifold"; if we say "a function", we mean "a smooth function"...

Once we understand how to define the smooth structure on M, the next natural question is

1.1.3 How to define the partial differential of a smooth function?

The naive idea is to take $\frac{\partial}{\partial x_k^{(j)}}(f \circ \phi_j^{-1})$ in the chart U_j as the partial differential $\frac{\partial}{\partial x_k} f$. In order to distinguish the partial differential in different charts, we use the notation $\frac{\partial}{\partial x_k^{(j)}}$ to represent the partial differential on the coordinate on $\phi_j(U_j)$. After all we know how to take partial differential on $\mathbb{R}^n$. We could do anything locally on $\mathbb{R}^n$. But the same problem appear. For another chart (U_i, ϕ_i) such that $U_i \cap U_j \neq \emptyset$, for $x \in \phi_j(U) \subset \mathbb{R}^n$, if $\frac{\partial}{\partial x_k^{(i)}}(f \circ \phi_i^{-1})$ and $\frac{\partial}{\partial x_k^{(j)}}(f \circ \phi_j^{-1})$ represent the same function $\frac{\partial}{\partial x_k} f$, from

(1.1.2), we need $\frac{\partial(f\circ\phi_j^{-1})}{\partial x_k^{(j)}}(x) = \frac{\partial(f\circ\phi_i^{-1})}{\partial x_k^{(i)}}(\phi_{ij}(x))$. However, by the chain's rule,

$$\frac{\partial(f\circ\phi_j^{-1})}{\partial x_k^{(j)}}(x) = \sum_{l=1}^{n} \frac{\partial(f\circ\phi_i^{-1})}{\partial x_l^{(i)}}(\phi_{ij}(x)) \cdot \frac{\partial \phi_{ij}^l}{\partial x_k^{(j)}}(x), \tag{1.1.4}$$

where ϕ_{ij}^l is the l-th component of the image of ϕ_{ij} in $\mathbb{R}^n$. This means that our naive idea is not compatible with our construction of the smooth structure. By (1.1.2) and (1.1.4), $\frac{\partial}{\partial x_k^{(j)}}(f\circ\phi_j^{-1})$ and $\sum_{l=1}^n \frac{\partial\phi_{ij}^l}{\partial x_k^{(j)}}(x)\frac{\partial}{\partial x_l^{(i)}}(f\circ\phi_i^{-1})$ represent the same function. Thus the partial differential

$$\frac{\partial}{\partial x_k^{(j)}} \text{ in } U_j \text{ corresponds to } \sum_{l=1}^{n} \frac{\partial \phi_{ij}^l}{\partial x_k^{(j)}}(x) \frac{\partial}{\partial x_l^{(i)}} \text{ in } U_i. \tag{1.1.5}$$

For the convenience, we could write (1.1.5) by matrix.

$$\begin{pmatrix} \frac{\partial}{\partial x_1^{(j)}} \\ \vdots \\ \frac{\partial}{\partial x_n^{(j)}} \end{pmatrix} \sim \left(\frac{\partial \phi_{ij}^l}{\partial x_k^{(j)}}(x)\right)_{(k,l)} \cdot \begin{pmatrix} \frac{\partial}{\partial x_1^{(i)}} \\ \vdots \\ \frac{\partial}{\partial x_n^{(i)}} \end{pmatrix}. \tag{1.1.6}$$

From this relation, we could glue the partial differentials in each chart together to get a global partial differential, which we call a vector field.

Definition 1.1.4. Let M be a manifold with smooth structure $\{(U_i,\phi_i)\}_i$. A vector field X on M is a collection of partial differentials $\sum_{k=1}^n a_k^{(j)} \frac{\partial}{\partial x_k^{(j)}}$, $a_k^{(j)}(x) \in \mathscr{C}^\infty(\phi_j(U_j)\subset\mathbb{R}^n, \mathbb{R})$, on each $\phi_j(U_j)\subset\mathbb{R}^n$ such that if $m \in U_i \cap U_j$,

$$a_l^{(i)}(\phi_i(m)) = \sum_{k=1}^{n} a_k^{(j)}(\phi_j(m)) \frac{\partial \phi_{ij}^l}{\partial x_k^{(j)}}(\phi_j(m)). \tag{1.1.7}$$

We often write $X|_{U_j} = \sum_{k=1}^n a_k^{(j)}\frac{\partial}{\partial x_k^{(j)}}$. Thus for a smooth function $f \in \mathscr{C}^\infty(M,\mathbb{R})$ and a vector field X, we could define Xf as a smooth function on M.

By definition, a vector field X is a map

$$X : \mathscr{C}^\infty(M,\mathbb{R}) \to \mathscr{C}^\infty(M,\mathbb{R}). \tag{1.1.8}$$

We must check the well-definedness of this definition. If $m \in U_i\cap U_j\cap U_s$, from (1.1.3), we have

$$\phi_{si}\circ\phi_{ij} = \phi_{sj}. \tag{1.1.9}$$

By chain's rule,

$$\frac{\partial \phi_{sj}^{l}}{\partial x_k^{(j)}}(\phi_j(m)) = \sum_{t=1}^{n} \frac{\partial \phi_{si}^{l}}{\partial x_t^{(i)}}(\phi_i(m)) \cdot \frac{\partial \phi_{ij}^{t}}{\partial x_k^{(j)}}(\phi_j(m)). \tag{1.1.10}$$

Thus from (1.1.7) and (1.1.10),

$$\begin{aligned} a_s^{(l)}(\phi_s(m)) &= \sum_{t=1}^{n} a_t^{(i)}(\phi_i(m)) \frac{\partial \phi_{si}^{l}}{\partial x_t^{(i)}}(\phi_i(m)) \\ &= \sum_{t=1}^{n}\sum_{k=1}^{n} a_k^{(j)}(\phi_j(m)) \frac{\partial \phi_{ij}^{t}}{\partial x_k^{(j)}}(\phi_j(m)) \frac{\partial \phi_{si}^{l}}{\partial x_t^{(i)}}(\phi_i(m)) \\ &= \sum_{k=1}^{n} a_k^{(j)}(\phi_j(m)) \frac{\partial \phi_{sj}^{l}}{\partial x_k^{(j)}}(\phi_j(m)). \end{aligned} \tag{1.1.11}$$

Thus our glue is compatible with the smooth structure on M.

In order to simplify the notations and perfect the theory, we want to find a home for the vector field, which we call the tangent bundle.

Let $\sqcup_i(U_i \times \mathbb{R}^n)$ be the disjoint union of $U_i \times \mathbb{R}^n$. We define an equivalent relation "$\sim$" such that $(x, (a_1^{(i)}, \cdots, a_n^{(i)})) \in U_i \times \mathbb{R}^n \sim (y, (a_1^{(j)}, \cdots, a_n^{(j)})) \in U_j \times \mathbb{R}^n$ if and only if $x = y$ and for any $1 \le l \le n$, $a_l^{(i)} = \sum_{k=1}^{n} a_k^{(j)} \frac{\partial \phi_{ij}^l}{\partial x_k}(x)$, i.e.,

$$(a_1^{(i)}, \cdots, a_n^{(i)}) = (a_1^{(j)}, \cdots, a_n^{(j)}) \cdot \left(\frac{\partial \phi_{ij}^{l}}{\partial x_k^{(j)}}(x) \right)_{(k,l)} \tag{1.1.12}$$

From (1.1.9) and (1.1.11), we see that this relation "$\sim$" is really an equivalent relation.

Definition 1.1.5. The tangent bundle is defined as the quotient space of the equivalent relation with the quotient topology, i.e., $TM := \sqcup_i(U_i \times \mathbb{R}^n)/\sim$.

Proposition 1.1.6. *The tangent bundle TM is a manifold.*

The proof is left to an exercise.

Let $\pi : TM \to M$ be the natural projection from $(x, v) \in TM$ to x.

Proposition 1.1.7. *A vector field is a smooth map $X : M \to TM$ such that $\pi \circ X = \mathrm{Id}$.*

The proof is left to an exercise.

A vector field is also called a section of TM. We denote by $\mathscr{C}^\infty(M, TM)$ the set of sections of TM.

1.1.4 What about the exterior differential?

Our natural idea is to take

$$d^{(j)}(f \circ \phi_j^{-1}) = \frac{\partial}{\partial x_k^{(j)}}(f \circ \phi_j^{-1}) \cdot dx_k^{(j)} \tag{1.1.13}$$

in the chart U_j as the exterior differential df. From the coordinate transformation formula in multivariable calculus, as in (1.1.6), we have

$$\left(dx_1^{(i)}, \cdots, dx_n^{(i)}\right) \sim \left(dx_1^{(j)}, \cdots, dx_n^{(j)}\right) \cdot \left(\frac{\partial \phi_{ij}^l}{\partial x_k^{(j)}}(x)\right)_{(k,l)}. \tag{1.1.14}$$

Equivalently, we have

$$\left(dx_1^{(j)}, \cdots, dx_n^{(j)}\right) \sim \left(dx_1^{(i)}, \cdots, dx_n^{(i)}\right) \cdot \left(\frac{\partial \phi_{ij}^l}{\partial x_k^{(j)}}(x)\right)_{(k,l)}^{-1}. \tag{1.1.15}$$

Thus from (1.1.6) and (1.1.15), we have

$$\begin{aligned} d^{(j)}(f \circ \phi_j^{-1})(x) &= \left(dx_1^{(j)}, \cdots, dx_n^{(j)}\right) \begin{pmatrix} \frac{\partial}{\partial x_1^{(j)}} \\ \vdots \\ \frac{\partial}{\partial x_n^{(j)}} \end{pmatrix} (f \circ \phi_j^{-1})(x) \\ &\sim \left(dx_1^{(i)}, \cdots, dx_n^{(i)}\right) \cdot \left(\frac{\partial \phi_{ij}^l}{\partial x_k^{(j)}}(x)\right)_{(k,l)}^{-1} \cdot \left(\frac{\partial \phi_{ij}^l}{\partial x_k^{(j)}}(x)\right)_{(k,l)} \cdot \begin{pmatrix} \frac{\partial}{\partial x_1^{(i)}} \\ \vdots \\ \frac{\partial}{\partial x_n^{(i)}} \end{pmatrix} (f \circ \phi_i^{-1})(\phi_{ij}(x)) \\ &= \left(dx_1^{(i)}, \cdots, dx_n^{(i)}\right) \cdot \begin{pmatrix} \frac{\partial}{\partial x_1^{(i)}} \\ \vdots \\ \frac{\partial}{\partial x_n^{(i)}} \end{pmatrix} (f \circ \phi_i^{-1})(\phi_{ij}(x)) = d^{(i)}(f \circ \phi_i^{-1})(\phi_{ij}(x)). \end{aligned} \tag{1.1.16}$$

Therefore, not like the partial differential, for the exterior differential, our naive idea is right. What we obtain is the following proposition.

Proposition 1.1.8. *The exterior differential defined d in (1.1.13) is globally defined.*

Like the vector field, we need to construct a home for df. We assume that on U_j, $df|_{U_j} = \sum_{k=1}^n b_k^{(j)} dx_k^{(j)}$. From (1.1.15),

$$df|_{U_i\cap U_j} = \left(dx_1^{(i)}, \cdots, dx_n^{(i)}\right)\begin{pmatrix} b_1^{(i)} \\ \vdots \\ b_n^{(i)} \end{pmatrix} \sim \left(dx_1^{(j)}, \cdots, dx_n^{(j)}\right)\begin{pmatrix} b_1^{(j)} \\ \vdots \\ b_n^{(j)} \end{pmatrix}$$
$$\sim \left(dx_1^{(i)}, \cdots, dx_n^{(i)}\right)\cdot\left(\frac{\partial \phi_{ij}^l}{\partial x_k^{(j)}}(x)\right)_{(k,l)}^{-1}\cdot\begin{pmatrix} b_1^{(j)} \\ \vdots \\ b_n^{(j)} \end{pmatrix}. \quad (1.1.17)$$

Thus

$$(b_1^{(i)}, \cdots, b_n^{(i)}) = (b_1^{(j)}, \cdots, b_n^{(j)})\cdot\left(\left(\frac{\partial \phi_{ij}^l}{\partial x_k^{(j)}}(x)\right)_{(k,l)}^{-1}\right)^T. \quad (1.1.18)$$

Definition 1.1.9. We define an equivalent relation "$\sim$" such that $(x, (b_1^{(i)}, \cdots, b_n^{(i)})) \in U_i \times \mathbb{R}^n \sim (y, (b_1^{(j)}, \cdots, b_n^{(j)})) \in U_j \times \mathbb{R}^n$ if and only if $x = y$ and (1.1.18) holds. The cotangent bundle is defined as the quotient space of this equivalent relation with the quotient topology, i.e., $T^*M := \sqcup_i (U_i \times \mathbb{R}^n)/\sim$.

From (1.1.5), the relation "$\sim$" is really an equivalent relation. Let $\pi' : T^*M \to M$ be the natural projection. As in the tangent bundle case, we denote by $\mathscr{C}^\infty(M, T^*M)$ the set of smooth maps $s : M \to T^*M$ such that $\pi' \circ s = \mathrm{Id}$. From the construction above, we have $df \in \mathscr{C}^\infty(M, T^*M)$. An element in $\mathscr{C}^\infty(M, T^*M)$ is also called a 1-form. Conversely, for any $\alpha \in \mathscr{C}^\infty(M, T^*M)$, there exists $b_k^i : \phi(U_i) \to \mathbb{R}$, $1 \le k \le n$ such that $\alpha|_{U_i} = \sum_{k=1}^n b_k^{(i)} dx_k^{(i)}$. From the knowledge of the multivariable calculus, there exists $f \in \mathscr{C}^\infty(\phi(U_i), \mathbb{R})$ such that $\alpha|_{U_i} = df$. Remark that this converse process is local. In general, for $\alpha \in \mathscr{C}^\infty(M, T^*M)$, we can not find $f \in \mathscr{C}^\infty(M, \mathbb{R})$ such that $\alpha = df$.

Furthermore, for $\alpha \in \mathscr{C}^\infty(M, T^*M)$, we can also define the exterior derivative of α: $d\alpha$. For

$$\alpha|_{U_i} = \sum_{k=1}^n b_k^{(i)} dx_k^{(i)}, \quad (1.1.19)$$

the natural idea to define $d\alpha$ by $d\alpha|_{U_i} = \sum_{l=1}^n \sum_{k=1}^n \frac{\partial b_k^{(i)}}{\partial x_l^{(i)}} dx_l^{(i)} dx_k^{(i)}$. As usual, we need to check the coordinate transformation. We simply denote by Φ the matrix $\left(\frac{\partial\phi^l_{ij}}{\partial x^{(j)}_k}(x)\right)_{(k,l)}$. From (1.1.6), (1.1.14) and (1.1.12), we have

$$b^{(i)}_k=\sum_{t=1}^{n} b^{(j)}_t\Phi^{-1}_{kt},\quad \frac{\partial}{\partial x^{(i)}_l}\sim\sum_{m=1}^{n}\Phi^{-1}_{lm}\frac{\partial}{\partial x^{(j)}_m},\quad dx^{(i)}_l\sim\sum_{q=1}^{n} dx^{(j)}_q\Phi_{ql}. \tag{1.1.20}$$

Thus

$$\begin{aligned}\sum_{l=1}^{n}\sum_{k=1}^{n}\frac{\partial b^{(i)}_k}{\partial x^{(i)}_l}dx^{(i)}_l dx^{(i)}_k &= \sum_{k,l,t,m,p,q=1}^{n}\Phi^{-1}_{lm}\frac{\partial(b^{(j)}_t\Phi^{-1}_{kt})}{\partial x^{(j)}_m}dx^{(j)}_q\Phi_{ql}dx^{(j)}_p\Phi_{pk}\\ &=\sum_{t,m=1}^{n}\frac{\partial b^{(j)}_t}{\partial x^{(j)}_m}dx^{(j)}_m dx^{(j)}_t+\sum_{t,m,p,k=1}^{n} b^{(j)}_t\frac{\partial(\Phi^{-1}_{kt})}{\partial x^{(j)}_m}\Phi_{pk}dx^{(j)}_m dx^{(j)}_p\\ &=\sum_{t,m=1}^{n}\frac{\partial b^{(j)}_t}{\partial x^{(j)}_m}dx^{(j)}_m dx^{(j)}_t-\sum_{t,m,p,k=1}^{n} b^{(j)}_t\Phi^{-1}_{kt}\frac{\partial^2\phi^k_{ij}}{\partial x^{(j)}_m\partial x^{(j)}_p}dx^{(j)}_m dx^{(j)}_p.\end{aligned} \tag{1.1.21}$$

The annoying term

$$\sum_{t,k=1}^{n} b^{(j)}_t\Phi^{-1}_{kt}\left(\sum_{m,p=1}^{n}\frac{\partial^2\phi^k_{ij}}{\partial x^{(j)}_m\partial x^{(j)}_p}dx^{(j)}_m dx^{(j)}_p\right) \tag{1.1.22}$$

appear. Note that $\frac{\partial^2\phi^k_{ij}}{\partial x^{(j)}_m\partial x^{(j)}_p}=\frac{\partial^2\phi^k_{ij}}{\partial x^{(j)}_p\partial x^{(j)}_m}$. A genius idea is that we define $dx^{(j)}_m dx^{(j)}_p=-dx^{(j)}_p dx^{(j)}_m$ to let (1.1.22) vanish. In order to avoid the ambiguity, for this anti-commutation property, we introduce a new notation: wedge product $\wedge$. And we use the notation $dx_m\wedge dx_p$ to replace $dx_m dx_p$ in the image of $d\alpha$. Here $dx_m\wedge dx_p$ means

$$dx_m\wedge dx_p=-dx_p\wedge dx_m. \tag{1.1.23}$$

Now we could define

$$d\alpha|_{U_i}=\sum_{l=1}^{n}\sum_{k=1}^{n}\frac{\partial b^{(i)}_k}{\partial x^{(i)}_l}dx^{(i)}_l\wedge dx^{(i)}_k. \tag{1.1.24}$$

From the arguments above, the definition in (1.1.24) does not depend on the choice of the coordinate. The next thing is to construct a home for the image of $d\alpha$. Now we do it in general. I'm tired to construct them one by one.

As in (1.1.23), we introduce the notation $dx_{p_1}\wedge\cdots\wedge dx_{p_k}$ satisfying

$$dx_{p_1}\wedge\cdots\wedge dx_{p_k}=\delta^{q_1,\cdots,q_k}_{p_1,\cdots,p_k}dx_{q_1}\wedge\cdots\wedge dx_{q_k}. \tag{1.1.25}$$

For $\alpha^{(i)}_{p_1,\cdots,p_k} \in \mathscr{C}^\infty(\phi(U_i),\mathbb{R})$, $1 \le p_1,\cdots,p_k \le n$, (1.1.19) is generalized to a k-form

$$\sum_{1\le p_1,\cdots,p_k\le n} \alpha^{(i)}_{p_1,\cdots,p_k} dx^{(i)}_{p_1} \wedge \cdots \wedge dx^{(i)}_{p_k} \tag{1.1.26}$$

on U_i. From (1.1.25), we arrange and restate (1.1.26) as

$$\sum_{1\le p_1<\cdots<p_k\le n} \beta^{(i)}_{p_1,\cdots,p_k} dx^{(i)}_{p_1} \wedge \cdots \wedge dx^{(i)}_{p_k}, \tag{1.1.27}$$

where $\beta^{(i)}_{p_1,\cdots,p_k} \in \mathscr{C}^\infty(\phi(U_i),\mathbb{R})$ and anti-commutes on $p_1,\cdots,p_k$. From (1.1.14), if $\sum_{1\le p_1<\cdots<p_k\le n} \beta^{(i)}_{p_1,\cdots,p_k} dx^{(i)}_{p_1}\wedge\cdots\wedge dx^{(i)}_{p_k}$ and $\sum_{1\le q_1<\cdots<q_k\le n} \beta^{(j)}_{q_1,\cdots,q_k} dx^{(j)}_{q_1}\wedge\cdots\wedge dx^{(j)}_{q_k}$ represent the same object on $U_i \cap U_j$, we can construct $(\Lambda^k\Phi)^{q_1,\cdots,q_k}_{p_1,\cdots,p_k} \in \mathscr{C}^\infty(\phi(U_j),\mathbb{R})$ such that

$$\beta^{(i)}_{p_1,\cdots,p_k} = \sum_{1\le q_1<\cdots<q_k\le n} (\Lambda^k\Phi)^{q_1,\cdots,q_k}_{p_1,\cdots,p_k} \cdot \beta^{(j)}_{q_1,\cdots,q_k}. \tag{1.1.28}$$

As in Definition 1.1.9, we define the bundle of exterior differentials.

Definition 1.1.10. We define an equivalent relation "$\sim_\Lambda$" on $\sqcup_i(U_i \times \mathbb{R}^{C^k_n})$ such that $(x,(\beta^{(i)}_{p_1,\cdots,p_k})_{1\le p_1<\cdots<p_k\le n}) \in U_i\times\mathbb{R}^{C^k_n} \sim (y,(\beta^{(j)}_{p_1,\cdots,p_k})_{1\le p_1<\cdots<p_k\le n}) \in U_j \times \mathbb{R}^{C^k_n}$ if and only if $x = y$ and (1.1.28) holds. The bundle of k-th exterior differentials is defined as the quotient space of this equivalent relation with the quotient topology, i.e., $\Lambda^k T^*M := \sqcup_i(U_i \times \mathbb{R}^{C^k_n})/\sim_\Lambda$.

Remark that if $k = n$, we have

$$\Lambda^n\Phi = (\det\Phi)^{-1}. \tag{1.1.29}$$

We could define $\mathscr{C}^\infty(M,\Lambda^kT^*M)$ as before. An element in $\mathscr{C}^\infty(M,\Lambda^kT^*M)$ is called a k-form on M. For $\alpha \in \mathscr{C}^\infty(M,\Lambda^kT^*M)$, $\alpha|_{U_i}$ could be written as (1.1.27). We define

$$d\alpha|_U = \sum_{1\le p_1<\cdots<p_k\le n} \sum_{t=1}^n \frac{\partial\beta^{(i)}_{p_1,\cdots,p_k}}{\partial x^{(i)}_t} dx^{(i)}_t \wedge dx^{(i)}_{p_1} \wedge \cdots \wedge dx^{(i)}_{p_k}. \tag{1.1.30}$$

As the same process in (1.1.20)-(1.1.24), we could obtain that $d\alpha$ is globally defined, which does not depend on the choice of the coordinate (try too fix it). Thus $d\alpha \in \mathscr{C}^\infty(M,\Lambda^{k+1}T^*M)$. Now we get a well-defined exterior differential

$$d : \mathscr{C}^\infty(M,\Lambda^kT^*M) \to \mathscr{C}^\infty(M,\Lambda^{k+1}T^*M). \tag{1.1.31}$$

Proposition 1.1.11. *(1)* $d^2 = 0$*;*

(2) for $\varphi \in \mathscr{C}^\infty(M)$*,* $d\varphi$ *is the one form such that* $\langle d\varphi, U\rangle = U(\varphi)$ *for any vector field* U*;*

(3) for any $\alpha \in \mathscr{C}^\infty(M, \Lambda^k T^*M)$*,* $\beta \in \mathscr{C}^\infty(M, \Lambda^l T^*M)$*, then*

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta. \tag{1.1.32}$$

The proof is left to an exercise. In fact, the exterior differential d is characterized by the properties in Proposition 1.1.11.

The differential forms and the exterior is very useful.

Theorem 1.1.12 (de Rham Theorem)**.** *If* M *is compact, the* k*-th cohomology of* M *with real coefficients*

$$H^k(M, \mathbb{R}) \simeq \frac{\mathrm{Ker}(d : \mathscr{C}^\infty(M, \Lambda^k T^*M) \to \mathscr{C}^\infty(M, \Lambda^{k+1} T^*M))}{\mathrm{Im}(d : \mathscr{C}^\infty(M, \Lambda^{k-1} T^*M) \to \mathscr{C}^\infty(M, \Lambda^k T^*M))}. \tag{1.1.33}$$

This theorem verifies our naive philosophy to get the global property from the local charts.

1.1.5 Differential operator on vector bundles

Now we generalize the tangent bundle, cotangent bundle, bundle of exterior differential in Definition 1.1.5, 1.1.9 and 1.1.10 to the vector bundle.

Definition 1.1.13. Let $\{U_i\}$ be an open covering of M. Let $m \in \mathbb{N}$. The transition function is a group of maps $\{\Psi_{ij} : U_i \cap U_j \to \mathrm{GL}(n, \mathbb{R})\}$ such that if $U_i \cap U_j \cap U_k \neq \emptyset$,

$$\Psi_{ki} \circ \Psi_{ij} = \Psi_{kj}. \tag{1.1.34}$$

We define an equivalent relation "$\sim$" on $\sqcup_i (U_i \times \mathbb{R}^m)$ such that $(x, (b_1^{(i)}, \cdots, b_m^{(i)})) \in U_i \times \mathbb{R}^m \sim (y, (b_1^{(j)}, \cdots, b_m^{(j)})) \in U_j \times \mathbb{R}^m$ if and only if $x = y$ and

$$(b_1^{(i)}, \cdots, b_m^{(i)}) = (b_1^{(j)}, \cdots, b_m^{(j)}) \cdot \Psi_{ij}(x). \tag{1.1.35}$$

A **vector bundle** with rank m is defined as the quotient space of this equivalent relation with the quotient topology, i.e., $E := \sqcup_i (U_i \times \mathbb{R}^m)/\sim$. As a manifold, E is called the total space and M is called the base space. We also have the natural projection (smooth) map $\pi : E \to M$. A (smooth) section of E is a (smooth) map $s : M \to E$ such that $\pi \circ s = \mathrm{Id}_M$. We denote by $\mathscr{C}^\infty(M, E)$ the set of sections.

If we replace $\mathbb{R}$ by $\mathbb{C}$ in Definition 1.1.13, we get a complex vector bundle.

Since the closure of U_i could be covered by coordinate charts, we always assume that U_i here is a coordinate chart.

Obviously, TM, T^*M and $\Lambda^k T^*M$ are vector bundles.

We could also construct new vector bundles from the old one. Let E and F be two vector bundles. We may assume that they are defined on the same covering $\{U_i\}$ (if not, take the common refinement). Let $\{\Psi_{ij}^E : U_i \cap U_j \to \mathrm{GL}(m, \mathbb{R}/\mathbb{C})\}$ and $\{\Psi_{ij}^F : U_i \cap U_j \to \mathrm{GL}(m', \mathbb{R}/\mathbb{C})\}$ be transition functions of E and F. Then we could construct

(1) $E \oplus F$, with transition function $\Psi_{ij}^E \oplus \Psi_{ij}^F$,

(2) $E \otimes F$, with transition function $\Psi_{ij}^E \otimes \Psi_{ij}^F$,

(3) E^*, with transition function $\left(\left(\Psi_{ij}^E\right)^{-1}\right)^T$,

(4) $\Lambda^k E$, with transition function $\Lambda^k \Psi_{ij}^E$ as in (1.1.28).

We usually denote by $\Lambda^\bullet E := \oplus_{k=1}^m \Lambda^k E$.

Obviously, $T^*M = (TM)^*$.

Now we generalize (1.1.30) and (1.1.31) to the differential operator.

We begin by fixing notation. For an n-tuple of nonnegative integers $\alpha = (\alpha_1, \cdots, \alpha_n)$, we set $|\alpha| = \sum_{i=1}^n \alpha_i$, and for each $\xi \in \mathbb{R}^n$, we set $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$. In local coordinates $(x_1, \cdots, x_n)$, we denote by $\frac{\partial^{|\alpha|}}{\partial x^\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$.

Definition 1.1.14. Let E and F be two complex vector bundles over M with $\operatorname{rank} E = p$ and $\operatorname{rank} F = q$. A **differential operator** of order m on M is a linear map $P : \mathscr{C}^\infty(M, E) \to \mathscr{C}^\infty(M, F)$ such that on each U_i,

$$P|_{U_i} = \sum_{|\alpha| \le m} A_\alpha^{(i)}(x) \frac{\partial^{|\alpha|}}{\partial x^{(i),\alpha}}, \tag{1.1.36}$$

where each $A_\alpha^{(i)}(x)$ is a $q \times p$-matrix of smooth functions and where $A_\alpha^{(i)} \neq 0$ for some α with $|\alpha| = m$.

By the knowledge of linear algebra, the complex matrix is easier to be handled than the real one. In the followings, we always assume that the differential operator acts on a complex vector bundle, except otherwise stated. For a real vector bundle, we first tensor it by $\mathbb{C}$, and then do the analysis.

We need to explain a bit about the right hand side of (1.1.36). For $s \in \mathscr{C}^\infty(M, E)$, on U_i, we could write $s|_{U_i} = \sum_{k=1}^p f_p s_p$, where $f_p \in \mathscr{C}^\infty(\phi(U_i) \subset \mathbb{R}^n, \mathbb{C})$ and $(s_1, \cdots, s_p)$ is a basis of $\mathbb{C}^p$. Consider a system of partial differential equations

$$\begin{cases} \sum_\alpha a_{11}^\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha} f_1 + \cdots \sum_\alpha a_{1p}^\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha} f_p = g_1, \\ \cdots \quad \cdots \quad \cdots \\ \sum_\alpha a_{q1}^\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha} f_1 + \cdots \sum_\alpha a_{qp}^\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha} f_p = g_q, \end{cases} \tag{1.1.37}$$

where $g_i \in \mathscr{C}^\infty(\phi(U_i) \subset \mathbb{R}^n, \mathbb{C})$. Note any system of PDEs could be written in the form of (1.1.37). Let $A_\alpha = (a_{ij}^\alpha)_{q\times p}$. We write (1.1.37) in the sense of matrix:

$$\sum_\alpha A_\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha} \begin{pmatrix} f_1 \\ \vdots \\ f_p \end{pmatrix} = \begin{pmatrix} g_1 \\ \vdots \\ g_q \end{pmatrix}, \tag{1.1.38}$$

which explains the right hand side of (1.1.36).

Easy to see that $d : \mathscr{C}^\infty(M, \Lambda^k T^*M \otimes \mathbb{C}) \to \mathscr{C}^\infty(M, \Lambda^{k+1} T^*M \otimes \mathbb{C})$ is a differential operator.

1.1.6 How to take derivative on the section of a vector bundle?

In Section 1.1.4, we define the exterior differential on a smooth function on M. In Definition 1.1.14, we see that it is a differential operator. If there is a vector bundle E over the manifold, the section is the natural generalization of the function. Is it possible to generalize the exterior differential d on the sections?

We repeat the process in (1.1.16) and (1.1.17) to see what is the same and what is not.

Let $s_k^{(i)} \in \mathscr{C}^\infty(U_i, \mathbb{C}^n)$ be the constant function on U_i such that for any $x \in U_i$, $s_k^{(i)}(x) = e_k$, $1 \le k \le m$. For any $s \in \mathscr{C}^\infty(M, E)$, there exist $f_k \in \mathscr{C}^\infty(U_i, \mathbb{C})$, $1 \le k \le m$, such that

$$s|_{U_i} = \sum_{k=1}^m f_k^{(i)} s_k^{(i)}. \tag{1.1.39}$$

From the definition of the vector bundle, on $U_i \cap U_j$,

$$(f_1^{(i)}, \cdots, f_m^{(i)}) = (f_1^{(j)}, \cdots, f_m^{(j)}) \cdot \Psi_{ij}(x). \tag{1.1.40}$$

Thus

$$\begin{pmatrix} s_1^{(j)} \\ \vdots \\ s_m^{(j)} \end{pmatrix} \sim \Psi_{ij}(x) \begin{pmatrix} s_1^{(i)} \\ \vdots \\ s_m^{(i)} \end{pmatrix}. \tag{1.1.41}$$

Naively, we try to define the exterior differential by $ds|_{U_i} = \sum_{k=1}^m (df_k^{(i)}) s_k^{(i)}$. The same problem appear.

$$\sum_{k=1}^m (df_k^{(i)}) s_k^{(i)} = (df_1^{(i)}, \cdots, df_m^{(i)}) \begin{pmatrix} s_1^{(i)} \\ \vdots \\ s_m^{(i)} \end{pmatrix}$$
$$\sim (df_1^{(j)}, \cdots, df_m^{(j)}) \begin{pmatrix} s_1^{(j)} \\ \vdots \\ s_m^{(j)} \end{pmatrix} + (f_1^{(j)}, \cdots, f_m^{(j)})(d\Psi_{ij}(x))\Psi_{ij}(x)^{-1} \begin{pmatrix} s_1^{(j)} \\ \vdots \\ s_m^{(j)} \end{pmatrix}. \tag{1.1.42}$$

The annoying term is $(d\Psi_{ij}(x))\Psi_{ij}(x)^{-1}$, which is a matrix of 1-form. But in this case, we don't have any idea kill it. Thus the exterior differential

$$d \quad \text{in } U_i \text{ corresponds to} \quad d + (d\Psi_{ij}(x))\Psi_{ij}(x)^{-1} \quad \text{in } U_j. \tag{1.1.43}$$

Remark that the matrix of 1-form $(d\Psi_{ij}(x))\Psi_{ij}(x)^{-1}$ acts on $s|_{U_j} = \sum_{k=1}^m (df_k^{(j)}) s_k^{(j)}$ by

$$(f_1^{(j)}, \cdots, f_m^{(j)})(d\Psi_{ij}(x))\Psi_{ij}(x)^{-1} \begin{pmatrix} s_1^{(j)} \\ \vdots \\ s_m^{(j)} \end{pmatrix}. \tag{1.1.44}$$

We will handle it by the same method as the vector field. We glue the exterior differentials by the transformation (1.1.43) together and give it a name: connection.

Definition 1.1.15. A (affine) connection ∇^E on E is a collection of $d + A^{(i)}$, where $A^{(i)}$ is a matrix of 1-forms, on each U_i such that on $U_i \cap U_j$,

$$A^{(j)} = A^{(i)} + (d\Psi_{ij}(x))\Psi_{ij}(x)^{-1}. \tag{1.1.45}$$

It is easy to see that ∇^E is a map

$$\nabla^E : \mathscr{C}^\infty(M, E) \to \mathscr{C}^\infty(M, T^*M \otimes E). \tag{1.1.46}$$

(Why? try to fix it.)

Recall that T^*M are the dual bundle of TM. Then for a vector field X and a 1-form α, we can define $\alpha(X) \in \mathscr{C}^\infty(M, \mathbb{R})$. In fact we've already used it in Proposition 1.1.11. Now we define

$$\iota_X : \mathscr{C}^\infty(M, T^*M) \to \mathscr{C}^\infty(M, \mathbb{R}), \quad \iota_X(\alpha) := \alpha(X). \tag{1.1.47}$$

Then by Proposition 1.1.11, we have

$$\iota_X \circ d = d : \mathscr{C}^\infty(M,\mathbb{R}) \to \mathscr{C}^\infty(M,\mathbb{R}). \tag{1.1.48}$$

Note that we can naturally define ι_X on $\mathscr{C}^\infty(M, T^*M \otimes E)$ by acting only on the T^*M part. Then we could define

$$\nabla^E_X := \iota_X \circ \nabla^E : \mathscr{C}^\infty(M,E) \to \mathscr{C}^\infty(M,E). \tag{1.1.49}$$

The operator ∇^E_X is usually regarded as taking the partial derivative on a section of a vector bundle along the direction X.

Exercise: Please check that ∇^E and ∇^E_X are all differential operators.

Proposition 1.1.16. *(1) For any $s_1, s_2 \in \mathscr{C}^\infty(M,E)$, we have*

$$\nabla^E(s_1+s_2) = \nabla^E s_1 + \nabla^E s_2. \tag{1.1.50}$$

(2) For any $s \in \mathscr{C}^\infty(M,E)$ and $f \in \mathscr{C}^\infty(M,\mathbb{C})$, we have

$$\nabla^E(fs) = (df)s + f\nabla^E s. \tag{1.1.51}$$

Proof. By definition. □

In many books, Proposition 1.1.16 is taken as the definition of the connection.

Remark that not like the exterior differential, the connection on the vector bundle is not uniquely defined. From Definition 1.1.15, locally, the difference of two connections is a matrix of 1-forms. Globally, the difference of two connections is a section of $\mathscr{C}^\infty(M, T^*M \otimes \mathrm{End}(E))$, where $\mathrm{End}(E) = E \otimes E^*$.

For a connection ∇^E on E and any $k \in \mathbb{N}$, there exists a unique extension $\nabla^E : \mathscr{C}^\infty(M, \Lambda^k T^*M \otimes E) \to \mathscr{C}^\infty(M, \Lambda^{k+1} T^*M \otimes E)$ verifying the Leibniz rule: for $\alpha \in \mathscr{C}^\infty(M, \Lambda^q T^*M)$, $s \in \mathscr{C}^\infty(M, \Lambda^{k-q} T^*M \otimes E)$, we have

$$\nabla^E(\alpha \wedge s) = d\alpha \wedge s + (-1)^q \alpha \wedge \nabla^E s. \tag{1.1.52}$$